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INVESTIGATION OF THE MIXED PROBLEM FOR THE SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. As is known, the Cauchy, boundary and mixed problems were mainly considered only for the certain systems of partial differential equations. The present paper is devoted to the investigation of the mixed problem for any system of partial differential equations. This system may be determined, over determined and sub-determined. A number of difficulties occur in investigation of these problems, some of which are of algebraic character.

Keywords: *definite, sub-definite and over determined systems, generalized solution, rectangular matrices, Laplace transformation.*

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1. Introduction

As it was mentioned above, only few papers were devoted to solving the problems for the both over-determined and sub-determined systems of the differential equations. These results are not complete not only for the differential equations, but also for the algebraic equations. Therefore, at first the algebraic character problems occur: how can a rectangular matrix be inverted? By this way, sometimes we reduce the general system to the normal form. It is interesting to note that the formulation of the mixed problem does not belong to the class of "usual" problems.

2. Problem statement

Consider the following mixed problem for the system of the first order linear differential equations of general form [3], [7]

$$A\frac{\partial u(x,t)}{\partial t}+B\frac{\partial u(x,t)}{\partial x}+Cu(x,t)=f(x,t),\ x\in(0,1),\ t>0, \tag{1}$$

$$\alpha u(0,t)+\beta u(1,t)=0, \quad t\geq 0, \tag{2}$$

$$Au(x,0)=A\varphi(x), \quad x\in[0,1], \tag{3}$$

where A, B and C are given matrices of dimension $(m\times n)$ with constant elements, $f(x,t)$ and $\varphi(x)$ are given matrices of dimensions $(m\times k)$ and $(n\times k)$, respectively, α and β are given matrices of dimension $(m\times n)$ and finally $u(x,t)$ is a desired matrix of dimension $(n\times k)$.

If $m = n$ (definite system), the matrices A, B and C may degenerate.
Let

$$rank\,B = rank\,(\alpha,\ \beta)\,. \tag{4}$$

Assuming that $u(x,t)$ is a solution, we apply the Laplace transform:

$$A\int_0^\infty e^{-\lambda t}\frac{\partial u(x,t)}{\partial t}dt + B\frac{d}{dx}\int_0^\infty e^{-\lambda t}u(x,t)dt + C\int_0^\infty e^{-\lambda t}u(x,t)dt = \int_0^\infty e^{-\lambda t} f(x,t)dt \tag{5}$$

Since,

$$\tilde{u}(x,\lambda) = \int_0^\infty e^{-\lambda t}u(x,t)dt\,, \tag{6}$$

$$\tilde{f}(x,\lambda) = \int_0^\infty e^{-\lambda t} f(x,t)dt\,, \tag{7}$$

integrating by parts the first term in (5) and taking into account the initial condition (3), we obtain

$$B\tilde{u}'(x,\lambda) + (\lambda A + C)\tilde{u}(x,\lambda) = \tilde{f}(x,\lambda) + A\varphi(x), \quad x \in (0,\, 1). \tag{8}$$

Here we assumed that

$$\lim_{t\to\infty} Ae^{-\lambda t}u(x,t) = 0\,. \tag{9}$$

In the same way, from boundary condition (2) one may obtain

$$\alpha\tilde{u}(0,\ \lambda) + \beta\tilde{u}(1,\lambda) = 0\,. \tag{10}$$

Thus, we obtain boundary value problem (8), (10) with parameters.

Remark 1. A.N. Tikhonov determined the solution of some problem as an element of the considered space [14]. At least, this solution gives minimum error. This problem may have no solution in the usual sense [14], [15].

Let's multiply system (8) from the left by the transposed matrix B, i.e.

$$\begin{aligned} &B^T B\tilde{u}'(x,\lambda) + (\lambda B^T A + B^T C)\tilde{u}(x,\lambda) = \\ &= B^T \tilde{f}(x,\lambda) + B^T A\varphi(x), \quad x \in (0,\, 1) \end{aligned} \tag{11}$$

We call the obtained system a transformed system.

In spite of the fact, that the matrices $B^T A$, $B^T B$ and $B^T C$ are quadratic, they may degenerate.

We represent system (11) in the following form [8]:

$$\begin{aligned} \tilde{u}'(x,\ \lambda) = \int_{-\infty}^0 e^{B^T B\tau}\{&B^T \tilde{f}(x,\lambda) + B^T A\varphi(x) - \\ &-(\lambda B^T A + B^T C)\tilde{u}(x,\lambda)\}d\tau = \\ = \int_{-\infty}^0 e^{B^T B\tau}B^T \tilde{f}(x,\lambda)d\tau &+ \int_{-\infty}^0 e^{B^T B\tau}B^T A\varphi(x)d\tau - \\ &-\int_{-\infty}^0 e^{B^T B\tau}(\lambda B^T A + B^T C)\tilde{u}(x,\lambda)d\tau\,, \end{aligned}$$

From this it is not difficult to obtain

$$\tilde{u}'(x,\lambda) = -\int_{-\infty}^{0} e^{B^T B\tau} d\tau(\lambda B^T A + B^T C)\tilde{u}(x,\lambda) + \\ + \int_{-\infty}^{0} e^{B^T B\tau} d\tau(B^T \tilde{f}(x,\lambda) + B^T A\varphi(x)) \tag{12}$$

Solving obtained system (12), we find

$$\tilde{u}(x,\lambda) = e^{-x\int_{-\infty}^{0} e^{B^T B\tau} d\tau(\lambda B^T A + B^T C)} \tilde{u}(0,\lambda) + \\ + \int_{0}^{\text{ч}} e^{-(x-\xi)\int_{-\infty}^{0} e^{B^T B\tau} d\tau(\lambda B^T A + B^T C)} \times \\ \times \int_{-\infty}^{0} e^{B^T B\tau} d\tau(B^T \tilde{f}(\xi,\lambda) + B^T A\varphi(\xi))d\xi\,. \tag{13}$$

Really, since

$$\tilde{u}'(x,\lambda) = -\int_{-\infty}^{0} e^{B^T B\tau} d\tau\ (\lambda B^T A + B^T C)\ e^{-x\int_{-\infty}^{0} e^{B^T B\tau} d\tau(\lambda B^T A + B^T C)} \tilde{u}(0,\lambda) + \\ + \int_{-\infty}^{0} e^{B^T B\tau} d\tau(B^T \tilde{f}(0,\lambda) + B^T A\varphi(x)) - \\ - \int_{-\infty}^{0} e^{B^T B\tau} d\tau(\lambda B^T A + B^T C)\cdot \int_{0}^{x} e^{-(x-\xi)\int_{-\infty}^{0} e^{B^T B\tau} d\tau(\lambda B^T A + B^T C)} d\xi \times \\ \times \int_{-\infty}^{0} e^{B^T B\tau} d\tau(B^T \tilde{f}(\xi,\lambda) + B^T A\varphi(\xi)),$$

it is easy to see that (13) is a solution of system (12).

Thus the following statement is proved.

Theorem 1. Let A, B and C be the given real matrices of dimension ($m \times n$), $f(x,t)$ and $\varphi(x)$ be given real-valued continuous matrices of the functions of dimension ($m \times k$) and ($n \times k$), respectively. If the matrix $B^T B$ is non-negative, then (13) is a solution of system (12).

Remark 2. System (8) may have no solutions. If in this case system (12) has a solution, then it will be called a generalized solution of system (8).

From (13) we easily get

$$\tilde{u}(1,\lambda) = e^{-\int_{-\infty}^{0} e^{B^T B\tau} d\tau(\lambda B^T A + B^T C)} \tilde{u}(0,\lambda) + \int_{0}^{1} e^{-(1-\xi)\int_{-\infty}^{0} e^{B^T B\tau} d\tau(\lambda B^T A + B^T C)} d\xi \times \\ \times \int_{-\infty}^{0} e^{B^T B\tau} d\tau(B^T \tilde{f}(\xi,\lambda) + B^T A\varphi(\xi))\,. \tag{14}$$

Substituting (14) into (10) we have

$$\alpha\tilde{u}(0,\lambda)+\beta\left\{e^{-\int_{-\infty}^{0}e^{B^TB\tau}d\tau(\lambda B^TA+B^TC)}\tilde{u}(0,\lambda)+\right.$$

$$+\int_0^1 e^{-(1-\xi)\int_{-\infty}^{0}e^{B^TB\tau}d\tau(\lambda B^TA+B^TC)}d\xi\times$$

$$\left.\times\int_{-\infty}^{0}e^{B^TB\tau}d\tau(B^T\tilde{f}(\xi,\lambda)+B^TA\varphi(\xi))\right\},$$

or

$$\left[\alpha+\beta e^{-\int_{-\infty}^{0}e^{B^TB\tau}d\tau(\lambda B^TA+B^TC)}\right]\tilde{u}(0,\lambda)=-\beta\int_0^1 e^{-(1-\xi)\int_{-\infty}^{0}e^{B^TB\tau}d\tau(\lambda B^TA+B^TC)}d\xi\times$$

$$\times\int_{-\infty}^{0}e^{B^TB\tau}d\tau(B^T\tilde{f}(\xi,\lambda)+B^TA\varphi(\xi)). \tag{15}$$

Thus, we obtain a system of the linear algebraic equations with rectangular matrices with respect to the matrix $\tilde{u}(0,\lambda)$. At first we represent it in the form

$$P(\lambda)\tilde{u}(0,\lambda)=Q(\lambda), \tag{16}$$

where

$$P(\lambda)=\alpha+\beta e^{-\int_{-\infty}^{0}e^{B^TB\tau}d\tau(\lambda B^TA+B^TC)}, \tag{17}$$

$$Q(\lambda)=-\beta\int_0^1 e^{-(1-\xi)\int_{-\infty}^{0}e^{B^TB\tau}d\tau(\lambda B^TA+B^TC)}d\xi\times$$

$$\times\int_{-\infty}^{0}e^{B^TB\tau}d\tau(B^T\tilde{f}(\xi,\lambda)+B^TA\varphi(\xi)). \tag{18}$$

Then we can represent equation (16) in the form

$$P^T(\lambda)P(\lambda)\tilde{u}(0,\lambda)=P^T(\lambda)Q(\lambda). \tag{19}$$

From this one may obtain

$$\tilde{u}(0,\lambda)=\int_{-\infty}^{0}e^{P^T(\lambda)P(\lambda)\xi}d\xi\, P^T(\lambda)Q(\lambda), \tag{20}$$

and $\tilde{u}(1,\lambda)$ is determined by means of relation (14).

Finally, the solution of spectral problem (8), (10) is found from (13).

Theorem 2. Under the conditions of Theorem 1 and (4), the generalized solution of mixed problem (8), (10) is represented in the form (13), where $\tilde{u}(0,\lambda)$ is found from (20).

The following statement is true for the solution of initial problem (1)-(3).

Theorem 3. Under the conditions of Theorem 2, the generalized solution of mixed problem (1)-(3) is represented in the form

$$u(x,t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{\lambda t}\tilde{u}(x,\lambda)d\lambda, \tag{21}$$

where $\tilde{u}(x,\lambda)$ is given in Theorem 2.

Using the schemes given in [1, 2, 4-6, 10], similar statements are proved for the multi-dimensional case.

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